1

Introduction to Modelling

Mathematical modelling, traditionally fundamental in Physics, has become a common practice in important branches of Ecology, Biology, Medicine and, even more relevantly, in Economics and Finance. This success relies on the special flexibility and the universality of mathematical tools, often capable of solving problems of great complexity. All this is made possible by the computing power of the increasingly sophisticated machines available today.

What is a mathematical model? How do we build it? A mathematical model is an interface between the real world and the world of mathematical theories. Consider, for instance, the price of a financial derivative and suppose we want find information on its evolution. If we want to use a mathematical model, it is necessary first of all to understand what are the factors (translated into mathematical variables) that we consider essential and characteristic of the evolution. This is probably the most delicate task. Then, we identify the fundamental relations between variables, that are capable to describe their dynamics quantitatively. Usually, this process results in several equations of various types, which constitute the mathematical model. The next step is to analyze the model, extracting the information we require.

What is the credibility of a mathematical model? It depends on its efficiency, which relies on the dichotomy between completeness and computability. Completeness, in general, would require the consideration of a huge number of factors and variables and the relationship between them, to be as realistic as possible; on the other hand, the greater the number of variables and the more realistic the relations among these variables, the more complicated the model is, putting at risk the computing capability. A good model realizes an efficient compromise, which always needs a posteriori test on known cases, to check its reliability. In any case, we have to keep in mind that a mathematical model (but any model describing real phenomena as well) cannot have any claim of universality and should be used *cum granu salis*. Some models allow quantitative analysis, for example the models in Physics, Chemistry, Medicine and Finance; in other models, typically those in Economics, the analysis can only be qualitative and it is basically used for testing the influence of certain factors on the evolution of a given system.

Why do we use mathematical models? Some of their characteristics make them particularly attractive: their low cost and flexibility, given the practically unlimited possibilities of running repeated computer simulations, in order to describe the trend of complex phenomena, otherwise incomprehensible. Sometimes mathematical models are the only instruments one can use; a clear example is the simulation of blood circulation in the so called *Willis Circle*, which guarantees the blood supply in both cerebral hemispheres. In this case, it is surely impossible to conduct experiments *in situ*!

Using a computer, one should always keep a critical attitude, avoiding the beginners naive declaration: "The computer said that, and so, that's it."

The models we will see in the first section are commonly used in applied science and are formed by *differential or difference equations or systems*. We will present the most elementary aspects of the theory (with some exceptions). Our aim is to teach the readers to be able to interpret and analyze the (differentials and difference) economics models they will encounter during their studies if not build on their own. Since this is conceived as an elementary course, we will deal with *deterministic* models only, where there are not *random* or *stochastic* terms.

1.1 Some Classical Examples

The mathematical models we will deal with are constructed by translating into mathematical terms some general laws of evolution, combined with specific laws of the phenomenon under description. We show the procedure using classical examples, that, at the same time, will help us to motivate the development of the theory in the next chapters.

1.1.1 Malthus model

Historically, this is the first model in *populations dynamics*, proposed by Malthus[1] in 1798. Consider an isolated population[2] whose only factors of evolution are *fertility* and *mortality*. We denote by $x(t)$ the number of individuals present at time t and we want to study its evolution from the starting (conventional) time $t = 0$. Obviously, $x(t)$ is an average value, which we can identify with a real number. Let λ be the fertility rate, that is the percentage of newly born individuals per unit of time (e.g. per year), and let μ be the mortality rate, that is the percentage of dead individuals per unit of time; thus, in a time interval of length h, the percentages of newly born individuals and dead individuals are, respectively, λh and μh.

Here the general law of evolution is simple: *the relative growth rate of the number of individuals in a time interval of live length h is a function of $(\lambda - \mu)\,h$.*

One has to choose *what kind of function to adopt*. Malthus assumes that this function is exactly $(\lambda - \mu)\,h$, so that, in mathematical terms, we have:

$$\frac{x\,(t+h) - x\,(t)}{x\,(t)} = (\lambda - \mu)\,h. \tag{1.1}$$

The number $\varepsilon = \lambda - \mu$ is called the *biological potential.*

[1] Thomas Robert Malthus (1766-1834), British economist.

[2] Not necessarily a population of human beings: it could be, for instance, a virus population.

We can now proceed with two different approaches, according to whether wish to follow the evolution of x continuously in time or at regular time intervals. The two approaches differ depending on whether we consider time as a *continuous* or a *discrete* variable, respectively.

Continuous time. In the first case, we divide equation (1.1) by h:

$$\frac{x(t+h)-x(t)}{h}=\varepsilon x(t),$$

and then we let $h \to 0$. We get:

$$\boxed{x'(t)=\varepsilon x(t)} \tag{1.2}$$

which says that the (relative) instantaneous growth rate (x'/x) is constant. Equation (1.2) is a *first order differential equation* since the unknown function x appears in the equation together with its *first derivative*. We say that it is a *linear* equation since it is a first degree polynomial in x and x'.

Can we solve (1.2) and determine the qualitative behavior of $x(t)$? Solving the differential equation means finding $x = x(t)$, defined at least for $t > 0$, which makes (1.2) an identity. In our case, we can also find the set of all solutions, i.e. the so-called *general integral*. Indeed, assume for the moment that $x(t) > 0$; then (1.2) can be rewritten as

$$\frac{x'(t)}{x(t)}=\frac{d}{dt}\ln x(t)=\varepsilon$$

whence, integrating both sides,

$$\ln x(t)=\varepsilon t+c,$$

where $c \in \mathbb{R}$ is an arbitrary constant. Applying the exponential function to both sides of the equality and letting $e^c = k$, we finally get

$$x(t)=e^{\varepsilon t+c}=e^c e^{\varepsilon t}\equiv ke^{\varepsilon t}. \tag{1.3}$$

Thus, the solution x has an *exponential growth* if $\varepsilon > 0$ and an *exponential decay* while $\varepsilon < 0$. When $\varepsilon = 0$, fertility and mortality balance and x is constant (figure 1.1).

The presence of the arbitrary constant k reveals that our model is not complete: to determine the evolution quantitatively we need an additional information, which in this case could be the *the number of individuals at the initial time*. Let us suppose that

$$x(0)=M_0>0. \tag{1.4}$$

We use this information, that we call *initial condition*, to determine the value of k in (1.3). If we set $t = 0$ in this equation, we find

$$M_0=k$$

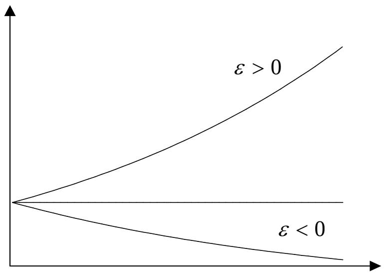

FIGURE 1.1. Exponential behaviour in the Malthus model

and then we have the complete behavior of the population at every time:

$$\boxed{x\,(t) = M_0 e^{\varepsilon t}} \tag{1.5}$$

From this formula we may deduce some interesting information. For example, if $\varepsilon < 0$, how long does it take for the population to halve its initial value? Does the halving time depend on the initial size?

Let T be the halving time. Then $x\,(T) = M_0/2$ and substituting in (1.5) we have

$$\frac{M_0}{2} = M_0 e^{\varepsilon T}$$

from which

$$T = -\frac{\log 2}{\varepsilon}.$$

Therefore the halving time does *not* depend on the initial population but only on its biological potential. We call this *population average life.*

Let us go back to (1.5) and suppose $\varepsilon > 0$. We point out that, if $M_0 > 0$, the solution never vanishes, since we divided by $x\,(t)$. On the other hand, if $x\,(t) \equiv 0$, we have again a solution of (1.2), corresponding to the zero initial condition: $M_0 = 0$. This solution is not negligible; indeed, since it is constant in time, it is called an *equilibrium solution.* As we can see in figure 1.1, when $\varepsilon < 0$, every other solution asymptotically goes to zero, independently of its initial condition: the population tends to extinction. If instead $\varepsilon > 0$, for every initial condition $M_0 > 0$, even if very small, the corresponding solution moves away from equilibrium: the population increases very rapidly beyond every limit. In the first case we will say that the equilibrium solution is *asymptotically stable*, in the second one that it is *unstable.*

The system

$$\begin{cases} x'\,(t) = \varepsilon x\,(t) \\ x\,(0) = M_0 \end{cases}$$

is called *Cauchy problem.* Our calculation says that, in this case, the solution exists for every t, it is unique and goes to zero or to infinity, as $t \to +\infty$, whether $\varepsilon < 0$ or $\varepsilon > 0$, respectively.

Finally, if we let the initial condition M_0 in (1.5), or equivalently, the constant k in (1.3), to vary among all real numbers we obtain the whole family of solutions, called the *general integral* of (1.2). We know everything about this simple (and important!) equation.

Discrete time. In the discrete time case, we check the evolution of the system at fixed time intervals. We use as a time unit the interval between two consecutive observations; then we have $t \in \mathbb{N}$. Thus the state of our system is described by a sequence $x(0)$, $x(1)$, $x(2)$, ... We write also x_0, x_1, x_2 and, in general, x_t instead of $x(t)$. Choosing $h = 1$ (the smallest possible interval) and placing it into (1.1), we have

$$x_{t+1} - x_t = \varepsilon x_t,$$

i.e.

$$\boxed{x_{t+1} = (1 + \varepsilon)\, x_t} \tag{1.6}$$

Equation (1.6) links the state of the system at time $t+1$ to the state at the previous time t; for this reason it is called a (*linear*) *one step* or *first order difference equation.* In this case too, the knowledge of the size of the population at the initial time $t = 0$ allows us to determine the unique solution:

$$\boxed{x_t = M_0\, (1 + \varepsilon)^t} \tag{1.7}$$

Indeed, letting $t = 0$ into (1.6), we get $x_1 = (1 + \varepsilon)\, M_0$ and, then, with $t = 1$, we have

$$x_2 = (1 + \varepsilon)\, x_1 = (1 + \varepsilon)^2\, M_0.$$

Iterating this procedure, we obtain

$$x_t = (1 + \varepsilon)\, x_{t-1} = (1 + \varepsilon)^2\, x_{t-2} = (1 + \varepsilon)^3\, x_{t-3} = \cdots = (1 + \varepsilon)^t\, M_0.$$

The zero solution corresponds to the initial condition $M_0 = 0$; this is the unique *equilibrium solution.* If we let the initial condition M_0 vary among all real numbers, we get the family of all solutions of the equation.

In this case, discrete and continuous-time systems behave likewise, and their solutions are described by a set of exponential functions; unfortunately, this is the exception and not the rule.

Even in this case the analogy between the two cases is not complete, since they exhibit some differences in the asymptotic behavior of the solutions. As in the continuous-time case, if $\varepsilon > 0$ then $1 + \varepsilon > 1$ and $(1 + \varepsilon)^t\, M_0 \to +\infty$ when $t \to +\infty$ (the zero solution is unstable). But, if $\varepsilon < 0$ things are different: if $\varepsilon > -2$ then $-1 < 1 + \varepsilon < 1$ and $(1 + \varepsilon)^t\, M_0 \to 0$ when $t \to +\infty$ (the zero solution is asymptotically stable); if $\varepsilon = -2$, then $1 + \varepsilon = -1$ and the solution is oscillating between $-M_0$ and M_0 (the zero solution is stable but not asymptotically stable), while if $\varepsilon < -2$, then $1 + \varepsilon < -1$ and x_t is oscillating and unbounded (the zero solution is unstable).

1.1.2 Logistic models

The Malthus model is too unrealistic. In this model, the external environment does not affect the growth rate and, consequently, the (relative) growth rate is steady. However, a bigger population entails fewer resources and this implies a smaller growth rate. In 1845, Verhulst[3] suggested a model in which the evolution law predicts a *survival threshold* M, which the population cannot exceed. He supposed that the relative growth rate of x, in the time interval h, linearly decreases as a function of x. Under these conditions, the evolution law can be written as:

$$x(t+h) - x(t) = \varepsilon h x(t)\left[1 - \frac{x(t)}{M}\right] \tag{1.8}$$

where $\varepsilon = \lambda - \mu$ is the biological potential. Let us distinguish again between continuous and discrete-time evolution.

Continuous time. We divide both sides of (1.8) by h and let $h \to 0$. We get:

$$x'(t) = \varepsilon x(t)\left[1 - \frac{x(t)}{M}\right]. \tag{1.9}$$

Equation (1.9) is a *nonlinear* first order *differential equation.* Although it is a little more complicated than (1.1) it is still possible to exhibit an explicit formula, representing the family of solutions and, in particular, the one satisfying the initial condition

$$x(0) = M_0.$$

We shall do this later on; here, we try to infer all possible information about the behavior of the solutions by analyzing the structure of the equation. It is useful to consider the differential equation as a relation converting the knowledge about the state of the system at time t, i.e. $x(t)$, into a growth rate (the slope of the graph of $x(t)$), i.e. $x'(t)$, at the same time. Observe that the two factors $\varepsilon x(t)$ and $[M - x(t)]/M$ are in competition: if $x(t)$ starts near zero, then $M - x(t) \sim M$ and $x'(t) \sim \varepsilon x(t)$, so that $x(t)$ increases exponentially. On the other hand, when $x(t)$ approaches M, the factor $M - x(t)$ becomes smaller and smaller and lowers the slope of $x(t)$, that becomes almost constant.

We observe that $x(t) \equiv 0$ is the *equilibrium* solution corresponding to the initial condition $M_0 = 0$. It is reasonable to think that it must be the unique solution starting from zero. However, there is another equilibrium solution: the one starting from the survival threshold M, and indeed, $x(t) \equiv M$ solves equation (1.9), as we see by direct substitution.

How do the other solutions behave? Do they tend to extinction? Do they tend towards the survival threshold? Do they tend to increase beyond every limit? In other words, we want information on the *asymptotic behavior* of the state variable for $t \to +\infty$ and in fact, this is what is required in many applications. To answer

[3]Pierre François Verhulst, 1804-1849.

these questions, we have to analyze

$$\lim_{t\to+\infty} x(t).$$

Assume $\varepsilon > 0$ and consider the solution starting from M_0, with $0 < M_0 < M$. From the differential equation, we have

$$x'(0) = \varepsilon x(0)\left[1 - \frac{x(0)}{M}\right] = \varepsilon M_0\left[1 - \frac{M_0}{M}\right] > 0$$

hence the solution leaves the initial state with a positive slope. Until $x(t)$ stays between 0 and M, $x'(t)$ remains positive and $x(t)$ is (strictly) increasing. Since monotone functions have limits, we deduce that only three possibilities can occur (figure 1.2).

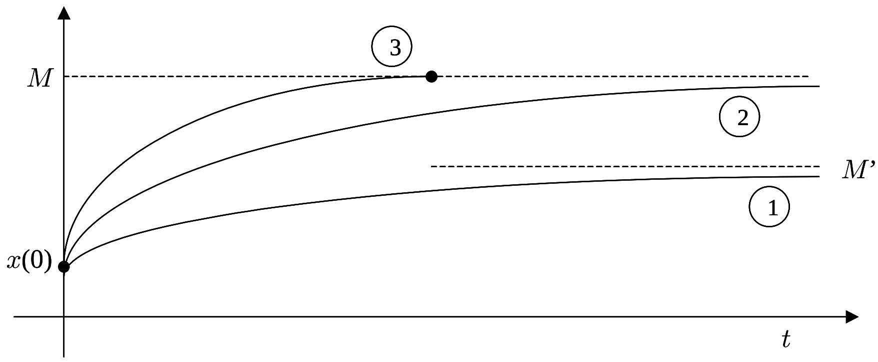

FIGURE 1.2. Logistic model: which number gives the correct evolution?

1. $\lim_{t\to+\infty} x(t) = M' < M$,

2. $\lim_{t\to+\infty} x(t) = M$,

3. $x(t)$ reaches the level M in a finite time T: $x(T) = M$.

We can immediately rule out case **1**. Indeed, since the limit of $x(t)$ is M', from the differential equation we would have

$$\lim_{t\to+\infty} x'(t) = \varepsilon M'\left[1 - \frac{M'}{M}\right] > 0. \tag{1.10}$$

But if a function and its derivative have *both* a horizontal asymptote, the limit of the derivative *has to be zero*[4], in contradiction to (1.10).

Case two seems to be the most reasonable: the population should tend to its threshold level.

[4] We point out that the derivative of a function which is strictly increasing in $(0, +\infty)$ and having a horizontal asymptote, may *have no limit*. But if we also know that its derivative has a limit, then the limit of the derivative must be zero. The reader can prove this as a useful exercise.

But can we exclude case **3**? We observe that if $x(t)$ reaches the level M in a finite time T, we have $x'(T) = 0$ and hence, thereafter x continues at the constant level M. There is no clear contradiction. Furthermore, if we use a computer with a time window sufficiently large, the answer would be the one sketched in figure 1.3, which seems to strengthen the possibility of case **3**.

Should we believe the computer or this case has to be explained in a less intuitive way? To answer we need to solve explicitly the equation or, better[5], use some of the theory we are going to develop later.

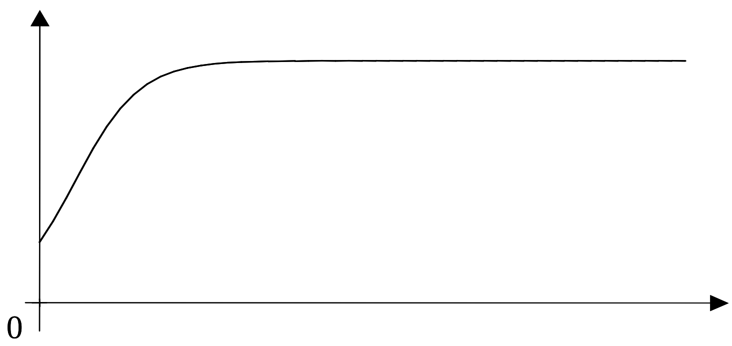

FIGURE 1.3. Computer graphic of continuous logistic with $r = 1$, $M = 2$, $M_0 = 0.5$

As an exercise, we leave the reader to study of the asymptotic behavior of a solution starting above the threshold level, i.e. such that $x(0) = M_0 > M$, as well as the case $M_0 < 0$.

Discrete time. We now consider the discrete logistic model. As before, choosing $h = 1$, from (1.8) we have

$$x_{t+1} = x_t + \varepsilon x_t \left(1 - \frac{x_t}{M}\right) = (1+\varepsilon)\, x_t \left(1 - \frac{\varepsilon}{(1+\varepsilon)\, M} x_t\right)$$

which is a *nonlinear difference equation* of the first order. To write the equation in a more readable form, let

$$x_t = \frac{(1+\varepsilon)M}{\varepsilon} y_t.$$

The equation for y_t is then

$$\boxed{y_{t+1} = r y_t \,(1 - y_t)} \tag{1.11}$$

where $r = 1 + \varepsilon$.

[5] As we shall see, there are only a few differential equations that can be solved with elementary techniques. Thus, in most cases, we have to use theoretical results to predict the behavior of the solutions and interpret graphs generated by computers.

The sequence $y_t = 0$ for every t (i.e. the sequence of zeros) is an *equilibrium solution.* What can we infer from (1.11) concerning the other solutions? First of all, is there another equilibrium solution? An equilibrium solution is constant, hence $y_{t+1} = y_t$. From (1.11) we get

$$y_{t+1} = ry_t\,(1 - y_t) = y_t$$

and

$$r\,(1 - y_t) = 1$$

from which

$$y_t = 1 - \frac{1}{r}.$$

This solution is between zero and one if $r > 1$ and therefore in this case there are two equilibrium solutions. Let us check if the other solutions tends towards one of the other two. If $-1 < \varepsilon < 0$ then $0 < r = 1 + \varepsilon < 1$, and if y_t starts from y_0 between *zero* and *one*, we immediately have

$$0 < y_{t+1} = ry_t\,(1 - y_t) < ry_t < r^2 y_{t-1} < \cdots < r^t y_0$$

and then $y_t \to 0$ per $t \to +\infty$. The zero solution is asymptotically stable.

Let now $\varepsilon > 0$. Then $r > 1$ and the two factors ry_t and $(1 - y_t)$ are in competition: the multiplication by r at each time step produces a rapid growth of y_t, but as soon as y_t approaches the level 1 the factor $(1 - y_t)$ becomes very small and produces the opposite effect. It is difficult to say *a priori* which one of the two factors is going to dominate. Also, we point out that $1-y_t$ could become negative and the sequence would lose its (physical) meaning of an average size of a population.

To get a clue of what could happen, we use a computer to plot the points (t, y_t) generated by the discrete logistic for different values of the parameter r. In figure 1.4, the plots are obtained with $y_1 = 0.2$, for r equal to 0.7 (circle), 2.6 (rhomb), 3.4 (cross), 4 (star).

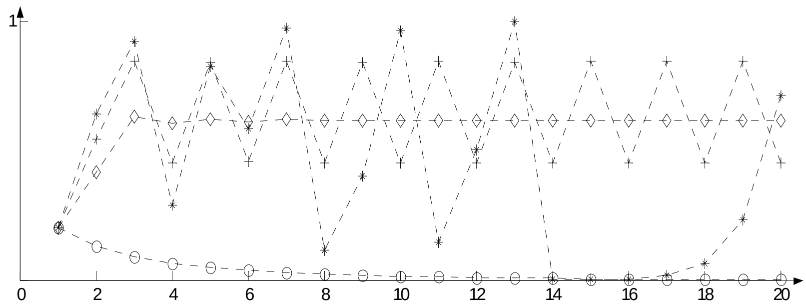

FIGURE 1.4. Sensitivity of the discrete logistic with respect to the parameter r

As it is quite apparent, the *discrete logistic* shows a strong dependence on the parameter r and rather unexpectedly a wide variety of asymptotic behaviors, including what is commonly called *chaotic behavior*. We shall describe later the theory necessary to interpret the graphs in figure 1.4.

1.1.3 Phillips model

We now consider a *continuous* time model of macroeconomic type, in which the level of the aggregate demand determines the national income. This demand should be created partly by private initiatives and partly by the government, whose aim is to reach a given level of national income.

Assume that we start at the desired level of income and, due to exogenous factors, a decrease in demand occurs. The problem is to stabilize the aggregate demand by means of suitable economic government policies, in order to restore the level of income.

To construct an evolution model, denote by $Y = Y(t)$ and $D = D(t)$, the *deviations* of the national income and of the aggregate demand from the desired levels, respectively. Since we are supposing that the initial product level is the correct one, we have $Y(0) = 0$.

The model is constructed on the basis of the following assumptions:

(i) national income responds to an excess of aggregate demand over supply by the law

$$Y' = \alpha(D - Y) \tag{1.12}$$

where $\alpha > 0$ is a sensitivity coefficient;

(ii) aggregate demand is a linear function of income

$$D = (1 - l)Y - u. \tag{1.13}$$

The *positive* constant u encodes the action of the exogenous factors, and $1 - l$ $(0 < l < 1)$ is the marginal propensity towards consumption and investment.

By substituting (1.12) into (1.11) we find that, without the government action, the evolution of Y is determined by the differential equation:

$$Y' = -\alpha lY - \alpha u, \tag{1.14}$$

with the initial condition $Y(0) = 0$. Equation (1.14) is *linear and of the first order.*

(iii) The fluctuations of the economy can be brought under control by varying the level of government expenditure G, every time this falls under some level G^*, i. e.

$$G' = \beta(G^* - G) \tag{1.15}$$

where $\beta > 0$ is a (constant) speed of adjustment. We examine two kinds of G^*:

(a) G^* is proportional to Y:

$$G^*(t) = -pY(t), \qquad (p > 0).$$

(b) G^* is proportional to the cumulative deficit of output below the desired level:

$$G^*(t) = -s \int_0^t Y(r)\,dr.$$

Anyway, instead of (1.13) and (1.14), we have

$$D = (1-l)\,Y - u + G$$

and

$$Y' = -\alpha l Y - \alpha u + \alpha G. \tag{1.16}$$

To get an equation in the unknown Y only, we differentiate both sides of (1.16); we find

$$Y'' = -\alpha l Y' + \alpha G'.$$

Using (1.15), we now have

$$Y'' = -\alpha l Y' + \alpha\beta\,(G^* - G)$$

and, gaining G from (1.16), we obtain the basic equation of the model

$$Y'' + (\alpha l + \beta)\,Y' + \alpha\beta l Y - \alpha\beta G^* = -\alpha\beta u, \tag{1.17}$$

where G^* depends on Y according to formulas (a) or (b).

Let us examine (1.17) in case (a). It becomes:

$$Y'' + (\alpha l + \beta)\,Y' + \alpha\beta\,(l+p)\,Y = -\alpha\beta u. \tag{1.18}$$

Since this equation is a first degree polynomial with respect to the unknown function Y and to its first and second derivatives, we say that (1.18) is a *second order linear* equation.

In case (b), (1.17) becomes

$$Y'' + (\alpha l + \beta)\,Y' + \alpha\beta l Y + \alpha\beta s \int_0^t Y(r)\,dr = -\alpha\beta u.$$

Differentiating in order to get rid of the integral, we get:

$$Y''' + (\alpha l + \beta)\,Y'' + \alpha\beta l Y' + \alpha\beta s Y = 0. \tag{1.19}$$

Equation (1.19) is a *third order linear* equation.

Some interesting information about this model concerns the qualitative behavior of a solution Y as $t \to +\infty$, depending on the parameters defining the model. We will devote special attention to linear differential equations, in particular to those with *constant coefficients,* as in Phillips model. For equations of this type it is also possible to find an explicit formula describing their general integral.

1.1.4 Accelerator model

This classical model (P.A. Samuelson, 1938) is an attempt to explain in a simplified way how expansion and recession cycles alternate in economic development. Consider an economic system observed over a number of consecutive years. Denote by Y_t the *national income* and by C_t the *consumption,* both in year t. The model is based on the following three laws:

(a) C_t is a linear function of the previous period's income:

$$C_t = aY_{t-1} + k, \tag{1.20}$$

where a and k are propensity coefficients, $0 < a < 1$ and $k > 0$.

(b) The *investment* I_t is a linear function of the consumption variations[6]:

$$I_t = A + b\left(C_t - C_{t-1}\right), \tag{1.21}$$

with $b > 0$.

(c) Y_t verifies the accounting identity (the level of production is chosen in order to meet the demand on goods)

$$Y_t = C_t + I_t.$$

Inserting I_t from (1.21) and C_t from (1.20), we get:

$$Y_t - a\left(1 + b\right)Y_{t-1} + abY_{t-2} = A + k. \tag{1.22}$$

Equation (1.22) is a difference equation linking Y_t to Y_{t-1} and Y_{t-2}. Since the left hand side is a first order polynomial in Y_t, Y_{t-1} and Y_{t-2} we say that (1.22) is a *second order linear equation* (or *a two steps equation*). We will solve it and study the evolution of Y_t as $t \to +\infty$ in chapter 5.

1.1.5 Evolution of supply

In this example we consider a simple model, which describes the price dynamics of a product and its supply. At equilibrium, demand equals supply. Denote by P the product price and by S the supply. Let X be the *excess in demand*, based on the condition at equilibrium. The evolution model is built on the following hypotheses.

(i) X is a linear function of the price:

$$X\left(t\right) = -x_0 + aP\left(t\right)$$

with $x_0 > 0$, $a > 0$.

(ii) Growth rate of supply is proportional to the excess of demand:

$$S'\left(t\right) = \lambda\left[-x_0 + aP\left(t\right)\right]$$

[6] The idea is the following: if for instance the consumer demand is growing, in order to meet the future demand, it is expedient to expand the production capacity. The opposite is to be done in case of contracting consumer demand.

with $\lambda > 0$.

(iii) Producers tend to direct $P(t)$ towards an optimal price $\widetilde{P}(t)$, according to the law

$$P'(t) = \beta\left[\widetilde{P}(t) - P(t)\right],$$

where $\beta > 0$. In this way, if $P(t) < \widetilde{P}(t)$ the price increases, while it decreases if $P(t) > \widetilde{P}(t)$.

(iv) Price $\widetilde{P}(t)$ is a linear function of supply:

$$\widetilde{P}(t) = M - bS(t) \tag{1.23}$$

with $M > 0$, $b > 0$.

Plugging (1.23) into the equation for P', we obtain the following evolution model for S and P:

$$\begin{cases} S'(t) = & a\lambda P(t) - \lambda x_0 \\ P'(t) = & -b\beta S(t) - \beta P(t) + M\beta. \end{cases} \tag{1.24}$$

This is a system of two linear differential equations in S and P[7].

Let us look for *equilibrium solutions*

$$\overline{S}(t) \equiv S^*, \quad \overline{P}(t) \equiv P^*.$$

Since $\overline{S}'(t) \equiv 0$, $\overline{P}'(t) \equiv 0$, substituting $\overline{S}(t) \equiv S^*$, $\overline{P}(t) \equiv P^*$ into the system (1.24), we find

$$\begin{cases} 0 = a\lambda P^* - \lambda x_0 \\ 0 = -b\beta S^* - \beta P^* + M\beta. \end{cases}$$

This is an algebraic system in the two unknown S^*, P^*, whose solution is

$$P^* = \frac{x_0}{a}, \quad S^* = -\frac{x_0}{ab} + \frac{M}{b}.$$

At this point the relevant questions are: what is the long time behavior of the other solutions? Do this model predict convergence to equilibrium or a *cyclic* behavior, with increases and decreases of price and supply?

We will answer to this questions in chapter 6.

[7] We point out that, differentiating the first equation, we get

$$S'' = a\lambda P'.$$

From the second equation and from $P = (S' + \lambda x_0)/a\lambda$, we obtain the second order equation in S:

$$S'' = -\beta S' - a\lambda b\beta S - \beta\lambda x_0 + a\lambda M\beta.$$

Thus, it is possible to transform the system into a second order equation in one of the two unknown functions only.

1.1.6 Leslie model

This is a model for a population growth that accounts for the different ages of individuals. Consider a population divided into disjoint age-sets of equal size. For instance, the population can be divided into three sets: if T is the oldest age, the individuals aged up to $T/3$ (*young individuals*) constitute the first set; the second set (*adults*) is composed by individuals aged between $T/3$ and $2T/3$, and the third set (*elder individuals*) by individuals over $2T/3$ years old. Choose $T/3$ as the time unit (time in which an individual is member of one of the three sets). Denote by

$$x_t, \quad y_t \quad \text{and} \quad z_t,$$

the number of individuals in the three sets at time t, respectively.

At time $t+1$, the "newly born" from every class enter the first class. Assume that the number of newly born in each class is proportional to the size of the class. In the second class we find the "survivors " belonging to the first class at time t and in the third the "survivors " belonging to the second class at time t. Let a, b and c be the birth-rate of the three classes, respectively, and α and β be the survival-rates of the first two classes (people in the third class cannot outlive the time interval). Thus the evolution of the population is governed by the following system:

$$\begin{cases} x_{t+1} = ax_t + by_t + cz_t \\ y_{t+1} = \alpha x_t \\ z_{t+1} = \beta y_t. \end{cases}$$

Denoting by $\mathbf{p}_t$ the vector of components x_t, y_t and z_t, the system may be rewritten in the form

$$\mathbf{p}_{t+1} = \mathbf{L}\mathbf{p}_t$$

where

$$\mathbf{L} = \begin{pmatrix} a & b & c \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix}$$

is called (third order) *Leslie matrix*.

This is a *linear homogeneous system of difference equations with constant coefficients*. We will study this kind of models in chapter 8.

In this case, it is important to deduce the long term behavior of the population. For instance, it is interesting to know whether the size of every class converges to a constant equilibrium value eventually.

1.1.7 Lotka-Volterra predator-prey model

This celebrated model deals with a survival problem for two different species living in the same habitat. One of them (the prey) constitutes the food supply for the other (the predator).

In the years immediately after the First World War, a decrease of a species of edible fish and an increase of its predator has been noticed in the Adriatic Sea waters. Unexpectedly, the absence of fishing activities during the war encouraged an above average growth of the predators, and caused the opposite effect on the prey. The question was put to the Italian mathematician Vito Volterra who solved the problem using a model, which constituted the first mathematical model in Ecology.

Volterra based his analysis on empirical information from biologists. The prey, whose main nourishment was constituted by microorganisms (phytoplankton), do not have survival problems, and their growth, without the presence of the predators, would follow a Malthusian model. The predators nutrition only consists of the prey. If isolated, their evolution would follow the Malthus model, but with a negative biological potential, so that they would decrease exponentially to zero. Denoting by

$$x(t) = \text{the (average) number of prey}$$

and by

$$y(t) = \text{the (average) number of predators,}$$

adopting a continuous-time model, we may write so far:

$$x'(t) = ax(t) + \ ? \quad \text{and} \quad y'(t) = -cy(t) + \ ? \qquad a > 0,\ c > 0.$$

The problem here is to model the interaction between the two species, assuming a negligible influence of the environment.

Volterra assumes that the main factor which slows down the growth of the prey and prevents the extinction of the predators is the (temporal) *frequency* of *encounters among prey and predators*. As a law for this frequency he adopts the following formulas, for prey and predators, respectively:

$$-bx(t)\,y(t) \quad \text{and} \quad dx(t)\,y(t) \qquad b > 0,\ d > 0.$$

Thus, he deduces the following model:

$$\begin{cases} x' = ax - bxy \\ y' = -cy + dxy. \end{cases} \qquad a, b, c, d > 0. \tag{1.25}$$

It is interesting to note that, more or less in the same period, the Polish chemist Lotka descovered the same model in relation to a problem of chemical kinetic, in a completely different contest. However, the analysis of Volterra was much deeper.

We now have a system of two equations, both *non-linear* with respect to the unknowns $x(t), y(t)$.

As in the supply evolution model, the relevant information is the following.

1. Are there *equilibrium solutions* (i.e. *constant* solutions)

$$\overline{x}(t) \equiv x^*, \quad \overline{y}(t) \equiv y^*?$$

Since $\overline{x}'(t) \equiv 0$, $\overline{y}'(t) \equiv 0$, letting $x(t) \equiv x^*$, $y(t) \equiv y^*$ in (1.25), we find

$$\begin{cases} 0 = ax^* - bx^*y^* \\ 0 = -cy^* + dx^*y^*. \end{cases}$$

This is an algebraic system solved by

$$(x^*, y^*) = (0,0) \quad \text{and} \quad (x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\right).$$

Thus, we have the two equilibrium solutions

$$\overline{x}(t) \equiv 0, \quad \overline{y}(t) \equiv 0,$$

which is not interesting, and

$$\overline{x}(t) \equiv \frac{c}{d}, \quad \overline{y}(t) \equiv \frac{a}{b},$$

which represents a steady coexistence solution.

2. What is the long term behavior of the solutions? Does the model predict the tendency to equilibrium or the extinction of one of the species? Or does it predict a *cyclic* trend with “ups and downs ” for prey and predators?

Even if this system can not be solved with elementary techniques, we will answer to these questions in chapter 7.

1.1.8 Time-delay logistic equation

A different form of the discrete logistic is the following second order equation

$$z_{n+1} = rz_n (1 - z_{n-1}), \qquad n \geq 1, \qquad r > 0.$$

The term z_{n-1} in the right-hand side denotes a *time delay* with respect to the original logistic model. It is convenient to transform the equation into a 2×2 system, setting

$$x_n = z_{n-1}, \qquad y_n = z_n.$$

Then we have

$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = ry_n (1 - x_n). \end{cases}$$

This is a system of two difference equations, which are quite difficut to study. The only information easy to obtain is the existence of equilibrium solutions, that leads solving the algebraic system

$$\begin{cases} x_n = y_n \\ y_n = ry_n (1 - x_n) \end{cases}$$

We find

$$(\overline{x}_n, \overline{y}_n) = (0,0) \quad \text{and} \quad (\overline{x}_n, \overline{y}_n) = \left(1 - \frac{1}{r}, 1 - \frac{1}{r}\right).$$

Observe that the second is meaningful only if $r > 1$. We will briefly examine the asymptotic behavior of the other solutions in chapter 8.

1.2 Continuous Time and Discrete Time Models

In the previous section we have considered mathematical models describing the time evolution of a variable, called *state variable*. In this section, we will analyze the structure of these models introducing also the appropriate terminology. We start with the case in which the state variable is *one-dimensional*.

Continuous time models. The time variable t runs over an interval of the real axis, which can often be identified with $[0, +\infty)$. In this case the function

$$t \mapsto x(t) \qquad t \in [0, +\infty),$$

describing the evolution of the state variable, is a real function of a real variable. As we have seen in the examples, x can represent the consistency of a population or the quantity of money available at a certain time, but the number of possibilities is countless.

Discrete time models. On many occasions, a periodic monitoring of time is more natural. In this case, if we use as time unit the time interval between two subsequent "observations", the variable t assumes integer values $0, 1, 2, \ldots$. The time unit in discrete models can be a year, a semester, a month, a day, etc.

The function describing the evolution of the state variable is a sequence, usually denoted by

$$t \mapsto x_t \qquad t \in \mathbb{N}.$$

Characteristic ingredients of a dynamical model are:

(a) the *dynamics,* i.e. the *evolution law* of x*,* that shows the connection between the state variable and its *growth rate*;

(b) the *initial condition,* i.e. the value of the variable x at the initial time t_0 (usually $t_0 = 0$).

The information about the *dynamics* of the system is expressed through an equation. In continuous time models we deal with *differential equations,* linking x to the values of one or few derivatives. Instead, in discrete time models, we deal with *difference equations*, linking the values of the state variable x at consecutive times. The simplest cases are sketched in the following table.

Model	*continuous time*	**dynamics**	$x'(t) = f(t, x(t))$
		initial condition	$x(0)$
	discrete time	**dynamics**	$x_{t+1} = f(t, x_t)$
		initial condition	x_0

The equation $x'(t) = f(t, x(t))$ is a *first order* differential equation, since (only) the first derivative of the state variable appears.

The equation $x_{t+1} = f(t, x_t)$ is a *first order* difference equation or a *one step difference equation*, since it connects the state at a time $t + 1$ to the state at the earlier time t. We expect that from (a) and (b) it is possible to deduce the entire evolution.

In the last examples of the previous section, the state variable is the *two-dimensional vector* $(x(t), y(t))$. In general, there are models describing the evolution of a n-dimensional state variable, the *vector state*

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

Also in this case we can distinguish between continuous and discrete models. In continuous models, the evolution is described by a vector valued function

$$t \mapsto \mathbf{x}(t)$$

where t belongs to an interval[8] $I \subseteq \mathbb{R}$. Instead, in discrete models, a sequence of vectors

$$t \mapsto \mathbf{x}_t, \quad t \in \mathbb{N},$$

describes the evolution.

The dynamics, i.e. the evolution law of the vector state $(x_1, x_2, \ldots, x_n)$, is expressed by a *system* of *differential equations* or *difference equations*, respectively. In the next section we establish a precise terminology.

1.2.1 Differential and difference equations

Differential equations

A *differential equation* is an equation where the unknown function appears together with some of its derivatives (the unknown function and its derivatives must be evaluated at the same point). The *order* of a differential equation is the order of the highest derivative appearing in the equation.

In an *ordinary differential equation* the unknown is a function of one variable, otherwise we have *a partial differential equation*[9]. A general *ordinary differential equation* (ODE) of order n has the form

$$F\left(t, x(t), x'(t), \ldots, x^{(n)}(t)\right) = 0 \tag{1.26}$$

where F is a function of $n+2$ real variables. The independent variable t represents time in dynamical models. In other types of model it could have different meanings, like space, for instance. As such it is usually convenient to adopt a more appropriate notation.

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[8] Usually the interval $[0, +\infty)$.

[9] E.g., the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

known as the *diffusion equation*, is a partial differential equation, since the unknown u is a function of the variables x e t.

If equation (1.26) can be rewritten in the form

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) \tag{1.27}$$

where f is a function of $n+1$ real variables, we say that the equation is in *normal form.* While we often write

$$x^{(n)} = f\left(t, x, x', \ldots, x^{(n-1)}\right)$$

we must remember that x and its derivatives are functions of time. We shall always refer to equations in normal form.

We give the precise notion of solution. A *solution* (sometimes also called *integral*, quite improperly) is a function $\varphi = \varphi(t)$, differentiable up to the order n, satisfying the equation in an interval $I \subseteq \mathbb{R}$, i.e. such that

$$\varphi^{(n)}(t) = f\left(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(n-1)}(t)\right) \qquad \forall t \in I. \tag{1.28}$$

This definition has a *local* nature, since φ is defined, a priori, in an interval, no matter how small it may be. On the other hand, an important question is to determine the largest possible interval of definition of a solution, to prepare for the analysis of its asymptotic behavior.

The *Cauchy problem* for a differential equation of order n in normal form consists of finding a solution of (1.27), satisfying the initial conditions

$$x(t_0) = x_1^0, \quad x'(t_0) = x_2^0, \ldots, \quad x^{(n-1)}(t_0) = x_n^0.$$

Linear equations represent an important class. Equation (1.27) is said to be *linear* if f is a *first degree polynomial* with respect to $x, x', \ldots, x^{(n)}$. For instance,

$$x' = tx + 1, \quad 3x'' - t^3 x' - x = 1, \quad x''' = (\log t)x - 3t^2 + 1$$

are linear equations, while the following ones are nonlinear:

$$x' = x^2, \quad x'' = \log x, \quad x''' = \sqrt{x'' + 4x' + 1.}$$

A general linear equation of order n has the form

$$a_0(t)\, x^{(n)} + a_1(t) x^{(n-1)} + \cdots + a_n(t) x(t) = b(t). \tag{1.29}$$

When $b(t) \equiv 0$ the equation is *homogeneous.* For linear equation, the structure of the family of solutions is known. If the coefficients a_j are constant, we can also write explicitly the family of all solutions (*general integral*), as we shall see in chapter 4.

One important distinction is between *autonomous* and *non autonomous equations.* Equation (1.27) is *autonomous* if f does not depend explicitly on t. We emphasize that the function f appearing in equation (1.27) is a function of $n+1$ independent variables, and, in general, t may be one of them. When t is not one of the explicit arguments of f, the equation is autonomous.

Malthus and Verhulst models involve autonomous equations. The following equations

$$x' = tx, \quad x'' = 2\sqrt[3]{t}x' + x + t$$

are non autonomous.

The difference between the two kinds of equation is easily seen by considering the two simple equations

$$x' = 3x \quad \text{(autonomous)} \quad \text{and} \quad x' = 3tx \quad \text{(non autonomous)}.$$

Suppose that at some time $\bar{t}$, $x = 10$. Can we deduce the growth rate x' from this information? In the first case, we immediately find that $x' = 30$, but in the second case, we need to know also at which *time* the state $x = 10$ is assumed: the information transfer

$$\text{state} \mapsto \text{growth rate}$$

depends only on the state for autonomous equations, while it depends on state and time for non-autonomous equations. As we shall see, this is an important difference.

Difference equations

The most general *k-steps difference equation* (or of *order* k) can be written as

$$F\left(t, x_t, x_{t+1}, \ldots, x_{t+k}\right) = 0 \tag{1.30}$$

where F is a function of $k + 2$ real variables. If

$$x_{t+k} = f\left(t, x_t, x_{t+1}, \ldots, x_{t+k-1}\right) \tag{1.31}$$

where f is a function of $k + 1$ variables, the equation is in *normal form.*

The *order* k is given by the difference between the highest and the lowest index appearing in the equation. For instance, the equation

$$x_{t+2} + 2x_{t+1} - tx_{t-1} + 1 = 0$$

is of order 3, since $t + 2$ is the maximum index, $t - 1$ is the minimum and their difference is 3.

A *solution* is a sequence $\{x_t\}$ satisfying the equation for every $t \in \mathbb{N}$ (or for every $t \geq t_0$, if t_0 is the first index). In general, we select a solution by fixing the initial data

$$x_0, x_1, ..., x_{k-1}$$

corresponding to the first k steps.

If F is a polynomial of degree one in $x_t, x_{t+1}, \ldots, x_{t+k}$, equation (1.30) is a *linear equation.* Some examples of linear equations are:

$$ax_{t+1} - bx_t = 1, \quad (\log t)x_t = tx_{t-1}$$

while

$$x_{t+1} = ax_t^2, \quad x_{t+1} - t\sqrt{x_t} + 2x_{t-1} = 0$$

are nonlinear equations.

If F in (1.30) or f in (1.31) is independent of t, the equation is *autonomous.*

Difference equations can be considered as the discrete version of differential equations. As we have done in introducing the Malthus and Verhulst models, we can always transform a differential equation of first order into a difference one, by replacing $x'(t)$ by its "discrete equivalent", i.e. the difference $x_{t+1} - x_t$. As a rule, we can transform a n^{th} order into a difference equation of order n, substituting:

$$\begin{array}{lll} x'(t) & \longrightarrow & x_{t+1} - x_t \\ x''(t) & \longrightarrow & (x_{t+2} - x_{t+2}) - (x_{t+1} - x_t) = x_{t+2} - 2x_{t+1} + x_t \\ x'''(t) & \longrightarrow & \begin{array}{l} x_{t+3} - 2x_{t+2} + x_{t+1} - (x_{t+2} - 2x_{t+1} + x_t) = \\ \quad = x_{t+3} - 3x_{t+2} + 3x_{t+1} - x_t \end{array} \\ \vdots & & \vdots \\ x^{(n)}(t) & \longrightarrow & \displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{t+n-k}. \end{array}$$

We remark, however, that discrete models usually display a much more complicated dynamics than the corresponding differential problems.

1.2.2 Systems of differential and difference equations

Suppose the time evolution of n real variables $x_1, x_2, \ldots x_n$ is governed by the *system of n first order differential equation* (in normal form):

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \ldots x_n) \\ x_2' = f_2(t, x_1, x_2, \ldots x_n) \\ \quad\vdots \\ x_n' = f_n(t, x_1, x_2, \ldots x_n). \end{cases} \tag{1.32}$$

Letting

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

where $\mathbf{f} : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$, the system can be rewritten in the vector form

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}). \tag{1.33}$$

This system expresses the *dynamics*, i.e. the *evolution law* of the *state vector* $\mathbf{x}$.

A *solution* of (1.32) in the interval I is a vector valued function

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}(t),$$

whose components are defined and differentiable in I and satisfy the equations of the system (1.32) for every $t \in I$.

As in the one dimensional case, the *value of the vector state* $\mathbf{x}$ *at a time* t_0 is often available in the applications. This leads to the *Cauchy problem*

$$\begin{cases} \mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases} \qquad \left(t_0 \in I,\ \mathbf{x}^0 \in \mathbb{R}^n\right).$$

obtained by coupling the *system of equations* with an *initial condition*.

If in (1.33) $\mathbf{f}$ is a linear function of the vector $\mathbf{x}$ the system is *linear*. A general linear system of n equations has the form

$$\begin{cases} x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + \cdots + a_{1n}(t)\, x_n + b_1(t) \\ x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + \cdots + a_{2n}(t)\, x_n + b_2(t) \\ \qquad\vdots \\ x_n' = a_{n1}(t)\, x_1 + a_{n2}(t)\, x_2 + \cdots + a_{nn}(t)\, x_n + b_n(t). \end{cases}$$

or

$$\mathbf{x}' = \mathbf{A}(t)\, \mathbf{x} + \mathbf{b}(t)$$

if

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

When $\mathbf{b}(t) = \mathbf{0}$, the system is said to be *homogeneous*. We will examine the general integral of linear systems in chapter 6.

When $\mathbf{f}$ does not explicitly depend on t, the system is *autonomous*. Chapter 7 is devoted to the study of two-dimensional autonomous systems.

A *system* of n one-step difference equation in normal form can be written as:

$$\begin{cases} x_1(t+1) = f_1(t, x_1(t), \dots, x_n(t)) \\ x_2(t+1) = f_2(t, x_1(t), \dots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(t, x_1(t), \dots, x_n(t)) \end{cases}$$

or in the vector forms

$$\mathbf{x}(t+1) = \mathbf{f}(t, \mathbf{x}(t)), \qquad \mathbf{x}_{t+1} = \mathbf{f}(t, \mathbf{x}_t) \tag{1.34}$$

where $\mathbf{f} : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$.

A sequence of vectors $\mathbf{x}_t$ is a *solution* of (1.34) if it satisfies (1.34) for every $t \in \mathbb{N}$.

If $\mathbf{f}(t, \mathbf{x})$ is a linear function of $\mathbf{x}$, the system is *linear*.

If $\mathbf{f}$ does not depend explicitly on time, the system is *autonomous*.

Reduction of a n-th order equation into a system of n first order equations

Every differential or difference equation of order n can be easily transformed into a system of n first order equations. For instance, in section 1.1, we reduced the discrete logistic equation with delay into a system of two difference first order equations.

In the continuous case, consider the differential equation of order n

$$x^{(n)} = f\left(t, x, x', \dots, x^{(n-1)}\right). \tag{1.35}$$

We can reduce it into a system of n first order equation by setting

$$x_1 = x, \quad x_2 = x', \quad \dots \quad x_n = x^{(n-1)}.$$

Then we obtain the system

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = f\left(t, x_1, x_2, \dots x_n\right). \end{cases} \tag{1.36}$$

The n-th order equation and the system are *equivalent*, in the sense that every *solution* $x(t)$ of (1.35) determine a vector

$$\left(x(t), x'(t), \dots, x^{(n-1)}(t)\right)$$

which is solution of (1.36) and, vice versa, the first component $x_1(t)$ of every solution $(x_1(t), x_2(t), \dots, x_n(t))$ of (1.36) is a solution of (1.35).

Example 2.1. Consider the linear non homogeneous differential equation with constant coefficients

$$x'' + 2\delta x' + \omega^2 x = f(t).$$

Setting $y_1 = x$ and $y_2 = x'$, the equation is transformed into the system

$$\begin{cases} y_1' = y_2 \\ y_2' = -\omega^2 y_1 - 2\delta y_2 + f(t), \end{cases} \tag{1.37}$$

or

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{f}(t)$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\delta \end{pmatrix} \quad \text{and} \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

We have here an example of non homogeneous *linear system with constant coefficients.*

Similarly, we can convert an n^{th} order difference equation into a system of n first order equations. For instance, consider the n^{th} order linear equation

$$x_{t+n} = a_n x_{t+n-1} + \cdots + a_2 x_{t+1} + a_1 x_t + b_t.$$

Setting

$$(x_t, x_{t+1}, \ldots, x_{t+n-1})^\top = \mathbf{y}_t = (y_1(t), y_2(t), \ldots, y_n(t))^\top \tag{1.38}$$

we get the system

$$\begin{cases} y_1(t+1) = y_2(t) \\ y_2(t+1) = y_3(t) \\ \qquad \vdots \\ y_n(t+1) = a_1 y_1(t) + a_2 y_2(t) + \cdots + a_n y_n(t) + b_t. \end{cases}$$

which may be rewritten in the vector form

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{b}_t$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_1 & a_2 & \ldots & \ldots & a_n \end{pmatrix}, \qquad \text{and} \quad \mathbf{b}_t = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_t \end{pmatrix}.$$

Example 2.2. Consider the equation

$$x_{t+2} - 2x_{t+1} + 2x_t = 0$$

and set

$$\begin{pmatrix} x_t \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} y_t \\ z_t \end{pmatrix}.$$

The equation is equivalent to the following *linear homogeneous system with constant coefficients*

$$\begin{cases} y_{t+1} = z_t \\ z_{t+1} = -2y_t + 2z_t. \end{cases}$$