FOREWORD TO THE SECOND EDITION

The first edition has been here thoroughly revised. We have duly eliminated misprints, a few mistakes and, in particular, a number of inconsistencies (hopefully, all of them) in the symbols used, as a result, the present text is definitively clearer.

We have also added a few arguments and remarks as well as several new exercises.

It was pretty hard going, believe us, but wellworth it.

We greatfully acknowledge M. Impedovo for helpful suggestions and for the careful rereading of the whole text. Finally we thank G. Barbieri for his robust english revision.

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The Authors

1 General questions

Probability is the proper existing tool for managing uncertainty. The fact that almost all the fields of Economics and of Management Science have to take uncertainty into account makes this tool necessary beyond any reasonable doubt. For instance, its relevance for Finance is ascertained and needs here no further remark.

However a correct use of probability is far from being generally diffused.

There are many reasons that can explain such a scarce diffusion, but, in our opinion, the main one is that most of the people facing uncertainty have frequently two opposite and extreme behaviors: some of them are terribly worried and anxious (and therefore take irrational decisions) whereas some others act too optimistically with a great self-confidence to be able to predict the future (and therefore take irrational decisions). Only few face uncertainty in a rational way.

One more reason making probability unfamiliar to many people is of logical nature. That there should be a number of different viewpoints about probability is a matter of course. In point of fact , the issue rapidly appears to be far more subtle than it looks like at first sight.

Take also into account the frequently observed confusion among:

- what probability is;
- how it can be assessed within special frameworks;
- how it can be manipulated, that is, how the arithmetic of probability works.

We shall deal with *complete theories* which will provide us with an answer to all the questions as well as an *incomplete theory*, which, popular though it might be, provides an answer only to the latter.

The historical sketch of the essence of probability we start out with is clearly meant to help the reader avoid the aforementioned confusion and keep in mind in which sense probability should be dealt with.

Consider two classical problems:

Example 1 [Oughtred rings problem] — *A box contains two rings. Maybe that both are golden (composition I) or only one is, the second being a silver one (composition II). You attribute the same probability ($\frac{1}{2}$) to both compositions. From the box you randomly extract a ring which turns out to be a golden one. The question is about the probability that the other ring should be golden too. Many people argue that the information conveyed by the first ring is far from decisive: the outcome "silver ring" would provide us with a certainty solution to the problem, whereas the "golden ring" outcome appears to leave things unchanged. Since the second ring can be a silver or a golden one, the majority of people think that the two compositions (I or II) continue to deserve the same probability. Well, we will see that this opinion is incorrect.*

Example 2 [Ten heads problem] — *A guy tosses a "perfect" coin. The outcome of every toss can be, with equal probability,* H*(ead) or* T*(ail). Ten tosses are made and the outcome is always* H*. The question is about the probability* p *to get* H *in the next toss. Some people say that* $p = \frac{1}{2}$*, because of the independence of the outcomes. Some people say, because of the law of large numbers and since so many heads occurred, that* p *should be smaller that* $\frac{1}{2}$ *(this party includes all the rather foolish people betting on the delays on Lotto results). Some people think that the coin might be not perfect as it should and, learning from experience, think, because of the observed sequence, that* $p > \frac{1}{2}$*. Clearly there are (apparent) justifications for all the possibilities, that is to believe that* $p = \frac{1}{2}$*,* $p < \frac{1}{2}$*,* $p > \frac{1}{2}$*!*

These examples show how questionable the field we are entering is. Our intuition, which may come in handy in many circumstances, can fail miserably when it comes to probability problems. Further examples will be provided to support our common inadequacy to cope with uncertainty contexts.

Human beings have tried to define probability depending on the way you approach every single problem. This fact constitutes a further source of difficulty, as the logical value of a statement is closely linked to the approach chosen.

The best way to cope with this problem is to review all of these possibilities (the so called "approaches to probability theory"), in order to understand their relevant characteristics which turn out to be:

- the practical context in which every approach was born and its range of applicability;
- the possible logical mistakes or confusions underlying each attempt to formalize a given notion of probability;
- the reasons for rejecting some of these conceptions stems from the fact that you may take into account the difference between two distinct questions, which are frequently intertwined and give thus rise to confusion: (1) what probability is; (2) how probability can be evaluated in specific problems.

We would very much like our reader to be confronted with a theory which should be both logically well-founded and in a position to help her/him solve problems of practical relevance[1].

Reviewing the various approaches we will realize that:

- the demand for a probability notion in a position to cope with a larger and larger set of problems has actually shown fallacies in the previously existing approaches, or, at least, strong practical limitations;
- the distinction between complete and incomplete theories does help reach a great clarity and a correct understanding of the various positions.

[1] We will try to keep the formal aspects at their lowest possible technical level, in order to make the presentation easy also for readers without specific mathematical training.